Investigation in Learning and Teaching of Graphs of Linear Equations in Two Unknowns

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Introduction

This article reports on a study conducted in 2011 for better understanding of students' learning difficulties on graphs of linear equations in two unknowns so as to improve the learning and teaching of this topic. It was the second part of a series of studies for improving teaching and learning of secondary mathematics based on in-depth analysis of assessment and pedagogy in selected topics. The study included testing and interviews with students, lesson design and tryout with teachers. Diagnostic tests and preliminary findings on students' misconceptions were informed by previous TSA results. Teaching and assessment materials were developed throughout the study and converted into teaching packages for online dissemination in Education Bureau's Web-based Learning and Teaching Support website (http://wlts.edb.hkedcity.net) after the study.

Curriculum and Basic Competencies

Knowledge about linear equations in 2 unknowns (or variables) and their graphs are essential for algebra learning after basic understanding and mastery of algebraic manipulations and equation solving. The concept is crucial for algebra learning in the later stage regarding solving systems of equations, functions and relations, coordinate geometry. In the current curriculum guide for Key Stage 3, this topic is subsumed under the unit of Linear Equations in Two Unknowns from the Number and Algebra dimension. The learning objective is described as “plot and explore the graphs of linear equations in 2 unknowns”.

In the common textbook treatments this objective may easily be interpreted as a pre-requisite of solving simultaneous equations with graphical methods, that is, considered mainly as a preparation for learning the techniques of solving equations. When the emphasis is put on the tools for solving equations, crucial concepts above equations in 2 unknowns and their graphical representations may not be properly developed. We find that the introduction to graphing of individual linear equations is provided in the beginning of the chapter on simultaneous equations. This section is usually followed by another on graphical method of solving equations where graphs of 2 equations will be plotted to determine their intersection. The usual practice on graphing is focused on setting simple tables of 3 columns, which is considered as an essential skill of getting appropriate points to determine a straight line on the graph paper manually.

More detailed specifications of the learning outcomes related to the learning objective are given by the following Basic Competency (BC) descriptors for the Territory-wide System Assessment (TSA):

NA13-1 Student can plot graphs of linear equations in 2 unknowns

NA13-2 Student can demonstrate recognition that graphs of equations of the form $ax + by + c = 0$ are straight lines

NA13-3 Student can determine whether a point lies on a straight line with a given equation

These learning outcomes can be considered as elaboration of the objective in the curriculum guide regarding plotting and exploring graphs of linear equations in 2 unknowns. These detailed specifications may help to shift the attention of the overall task of plotting 2 equations from getting the intersection and extracting the solution to both equations. It is now more likely to observe what the students can or cannot do when simply plotting one equation and interpret any point on line as possible solution to an equation in 2 unknowns.

Performance in TSA

The BC "NA13-1: Student can plot graphs of linear equations in 2 unknowns" seems to expect a very simple skill of graphing a single equation (compared with coordinating graphs of two equations and getting the intersection). However, students' performances over the years are consistently low. In 2008 and 2009, there were parallel questions in separate papers asking students to plot graphs with and without providing a table. Results in these two years obviously show that the questions without table are answered less as well (Figure 1).

<u>2008 TSA Paper 1 Q.34</u> *Facility: 34.8%*

Draw the graph of $x + y = 1$ on the given rectangular coordinate plane in the ANSWER BOOKLET.

<u>2008 TSA Paper 2 Q.49 & Paper 3 Q.48</u> *Facility: 76.2% (table); 68.4% (graph)*

Complete the following table for the equation $2y = x + 4$ in the ANSWER BOOKLET:

x	-4	0	4
y			4

Draw the graph of this equation on the rectangular coordinate plane given in the ANSWER BOOKLET.

<u>2009 TSA Paper 1 Q.28</u> *Facility: 30.9%*

Draw the graph of $2y = x + 1$ on the rectangular coordinate plane given in the ANSWER BOOKLET.

<u>2009 TSA Paper 2 Q.43 & Paper 3 Q.43</u> *Facility: 71.6% (table); 61.6% (graph)*

Complete the following table for the equation $2y = x + 1$ in the ANSWER BOOKLET.

x	-3	0	3
y			2

Draw the graph of this equation on the rectangular coordinate plane given in the ANSWER BOOKLET.

Figure 1

Evaluation of random sample answer scripts on these questions also reveals qualitatively different answers to these questions. That means removing the table from the questions may not simply increase the procedural or computational difficulty of the tasks and result in more mistakes. It may also suggest that students can interpret the task in another way and show their lack of fundamental understanding of the graphing activity. For example, the answer in Figure 2 is found in 6 answer scripts among 100 randomly selected scripts containing 2009 Paper 1 Q28. In this figure, exactly one point (1,2) is marked but it is not clear how this is related to the given equation. Another 16 answers, out of those 100 sample scripts, are also found to be similarly incomprehensible. However, from sample scripts of other papers containing the parallel question with table, no such kind of answers is found.

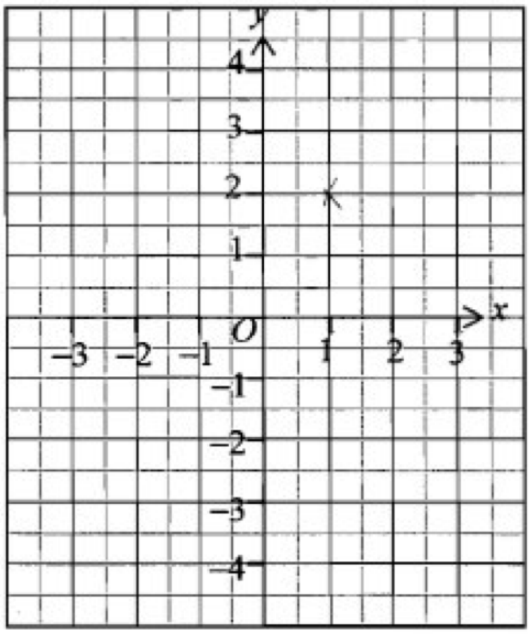


Figure 2

On the other hand, another kind of rather common responses are found among those wrong answers associated with tables. Since mistakes in calculation there may lead to 3 non-collinear points, students may join the 3 points to make a triangle in such case (like the example show in Figure 3). It is noteworthy that these cases suggest clearly students' misinterpretation of the procedure of joining the points as creating a geometric figure rather than producing a representation of infinite solutions to an equation.

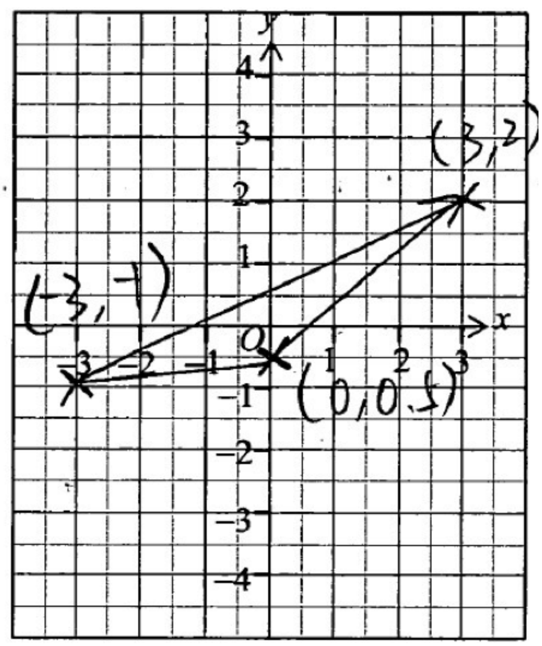


Figure 3

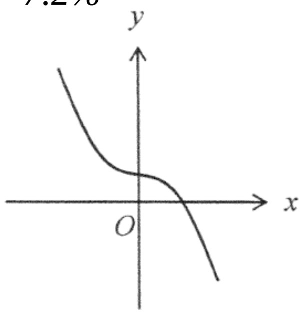
The BC "NA13-2: Student can demonstrate recognition that graphs of equations of the form ax + by + c = 0 are straight lines" focuses on recognition of the distinction between linear and non-linear graphs in terms of their appearance as a straight line or other forms. Such conceptual understanding was not explicitly emphasized in the previous curriculum. This characteristic of linear graphs should be accepted by students if they have to handle confidently the graphical method of solving equations. However, little is known about how they acquire this concept from their limited experience working with equations and graphs. On the other hand, it is also not clear to what extent and how this concept should be explained to students. In fact, a satisfactory explanation may involve deeper understanding of graphs and functions, which is usually not expected in this stage.

While identifying this subtle difficulty in teaching and learning, questions on this learning outcome in the papers so far can barely assess students' understanding in this area. The form of questions is very similar in all these papers. Students are asked to identify from 4 given graphs the one for a given linear equation. They need not pay attention to the details of the graphs and the major difference among the options is the linearity. The following item in 2006 illustrates the type of questions asked similarly in the subsequent years.

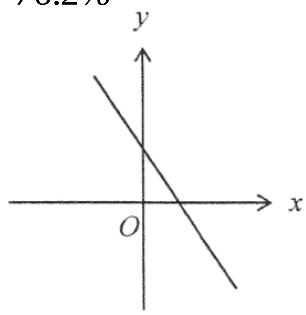
<u>2006 TSA Paper 3 Q.4</u>

Which of the following may represent the graph of the equation $3x + 2y - 1 = 0$?

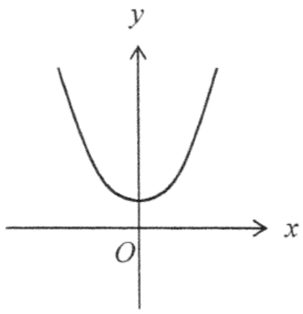
○ A. *7.2%*



○ B. *76.2%*



○ C. *5.2%*



○ D. *10.4%*

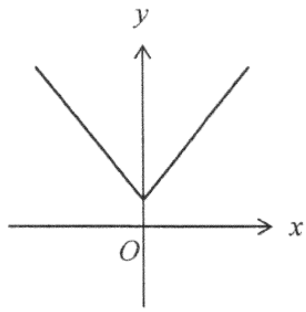


Figure 4

The facilities of these questions from 2006 to 2010 are respectively 76.2%, 74.2%, 76.3%, 74.1% and 69.9%. There is obviously limitation on this way of asking. The given equation must be linear since they are not expected to recognize or have experience on other types of equations. Among all the options, all they have to choose is a linear graph and most probably they have not worked with other types of graphs of equations. Despite this repeated form of questions, students' performance remains in a relatively stable level of 75% facility.

As suggested from these results, we have some crucial questions about teaching and learning.

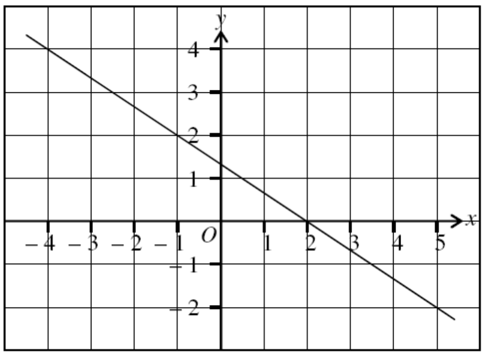
- Why do students fail to recognize this apparently obvious characteristic of linear graph?
- Does the failure to recognize this characteristic interfere with the general work of graphing and solving equations?

- Is this inability to distinguish linear graphs linked with other fundamental conceptual obstacles of graphing a general or only a specific issue about linear equations?
- For students who answer correctly, will they merely answer by eliminating unfamiliar examples or relying on superficial clues?

Some of these become guiding questions in the subsequent investigation.

The BC "NA13-3: Student can determine whether a point lies on a straight line with a given equation" is also closely related to students' understanding of the meaning of solutions to an equation of 2 unknowns, whether verified algebraically or represented graphically. There are two main types of questions. More often, a linear equation and its graph are given while students are asked to decide whether given points are solutions to the equation or lying on the graph. This can be done partially by quick inspection on the graph and further verified by substitution in the equation. In other questions, the graph is not given and students can only verify a solution by algebraic substitution (Figure 5). Facilities of these questions in 2006 to 2010 range from 41.8% to 67.4%. It is quite clear that the performance remains in a relatively low level.

<u>2006 TSA Paper 3 Q.4</u> *Facility: 50.9%*



The figure above shows the graph of $2x+3y=4$. Which of the following points lie on the straight line? (There may be more than one answer.)

$P(-3\ ,3)$, $Q(-1\ ,2)$, $R(0\ ,1)$, $S(4\ ,-\frac{4}{3})$

Figure 5

Pretest and Diagnosis

A learning study was carried out in a secondary school. Before the study, a diagnostic test was conducted in the school. The test covers some key learning objectives related to linear graphs, partly based on the findings explained in the previous section. It is intended to explore understanding of secondary 3 students who have just studied the topic of simultaneous equations, including the knowledge of graphical methods. It is also considered as a pretest to the following learning study. Due to the special arrangement of the school in this year, this topic will also be taught to secondary 2 students in the second term. Findings from the test can therefore inform our design of the research lesson on the same topic.

The test starts with several questions directly copied from the TSA. These include questions for plotting graphs with or without a table given (the 2009 TSA question in Figure 1). It also includes multiple-choice questions for distinguishing linear from non-linear graphs (e.g. the 2006 TSA question in Figure 2). The rest of the test is further developed from similar questions in order to gather more detailed information about students' understanding in this topic. Several of these questions require students to compare algebraic relations expressed in the form of symbolic equations, tables and graphs (Figure 6 and 7). They are also required to distinguish points that belong to an equation or not while presenting these values of *x* and *y* in different ways (Figure 8). Another type of questions require students to distinguish linear and non-linear equations (Figure 9).

Which of the following equations is satisfied by the data in the table?

x	-2	-1	0	1
y	-5	-4	-3	-2

☐ A. $y = x - 3$
☐ B. $y = x - 1$
☐ C. $y = 2x - 1$
☐ D. $y = 3x + 1$

Figure 6

Which of the following sets of data can be used to draw a straight line on the rectangular coordinate plane? (You can choose more than one answer.)

☐

x	y
-1	18
0	20
1	22
2	24

☐

x	y
-1	1
0	3
1	7
2	11

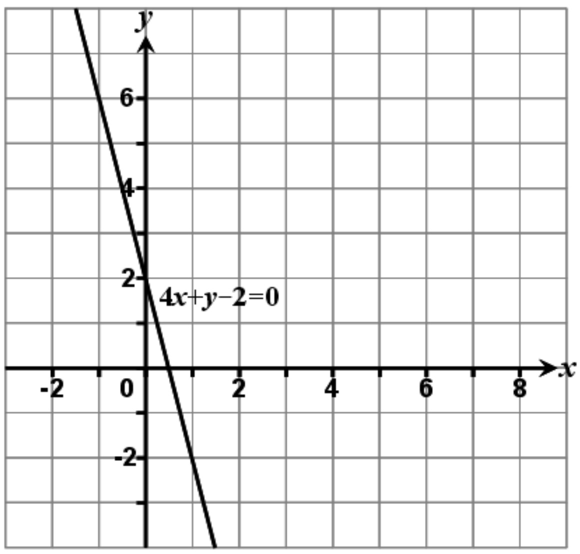
☐

x	y
-1	2
0	3
1	8
2	16

☐

x	y
-1	50
0	45
1	40
2	35

Figure 7



The figure above shows the graph of $4x + y - 2 = 0$. (Put a tick (✓) into the box of the answer.)

Point	Does the graph of $4x + y - 2 = 0$ pass through the point?
$(-1,\ 6)$	☐ Yes ☐ No
$(1,\ 1)$	☐ Yes ☐ No
$(-2,\ 10)$	☐ Yes ☐ No
$(-5,\ 18)$	☐ Yes ☐ No
$(-2,\ -6)$	☐ Yes ☐ No
$(a,\ 2-4a)$	☐ Yes ☐ No

Figure 8

Put ticks (✓) into the boxes of the following equations whose graphs are straight lines.

☐	$y = 2x$	☐	$\frac{1}{2}x = 3y$
☐	$x = 2 - 3y$	☐	$y = \frac{1}{1+x^2}$
☐	$2x + 3y - 5 = 0$	☐	$y = -1$
☐	$y = x^2$	☐	$y = \frac{1}{x}$
☐	$xy = 4$	☐	$y = \frac{1}{2-x}$

Figure 9

The test was done by 5 classes in secondary 3. The following are some major findings and questions about students' learning from analysis of their answers.

- Similar to the results of TSA, when a table is not provided, performance on graph plotting is significantly poorer.
- Some students merely join the points in a graph with line segments only, instead of a line extended to the edges of the graphing region. Do they notice any difference between lines and line segments for representing an equation?
- Most students can carry out substitution for finding values in an equation. However, this skill may not lead to understanding of the relation between equations and graphs.
- When matching a graph with an equation, some students may evaluate only one pair of values. Do they know that checking one ordered pair is not sufficient?
- When they are given tables of values but without equation, it is not easy to identify linear relations between the variables.
- When they are asked to suggest equations containing a specific point, most can provide only one such equation. Do they know that many different equations can have the same point lying on their graphs?

The diagnostic test is followed by task-based interviews. Students are selected based on their responses to the test. During the interviews, students are asked to repeat some questions from the test and explain to interviewers their reasoning. An additional group interview is also arranged to test students' responses when they are working together on a learning task that may be done in normal classroom setting.

The individual interviews show that the students are generally capable of checking individual points on a graph by substitution in an equation. Therefore, they basically understand the relation between a point on a graph and a solution for an equation. However, such knowledge may not sufficiently support their work and reasoning on various unfamiliar tasks relating equations, tables and graphs. One non-trivial task for them is to distinguish between linear and non-linear equations or graphs. When they are given some graphs with various forms for matching with a linear equation, they may not immediately recognize that the graph should be a line. Instead, they may refer to the basic principle by testing sample points or rely on a standard plotting procedure to find out which graph is appropriate (Figure 10). This shows that these students cannot readily accept this assumption that linear equation should generate a line graph and in fact this assumption may not be thoroughly explained due to limitation from the curriculum and common classroom practice.

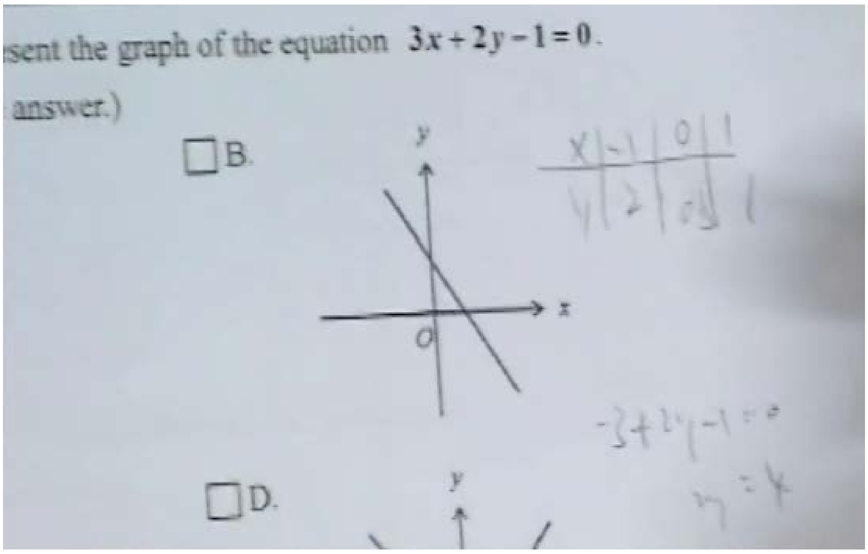
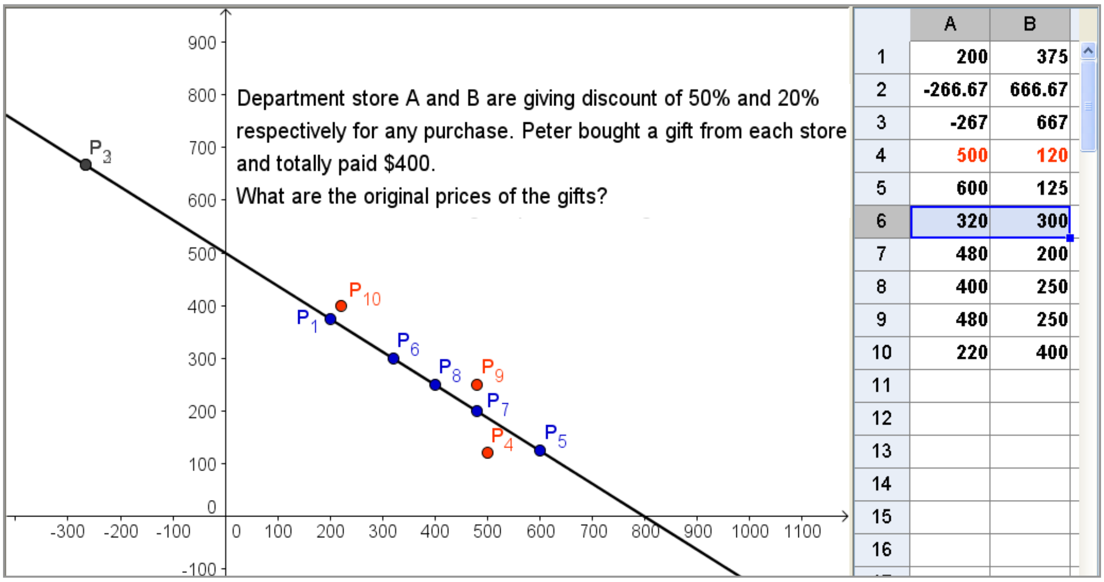


Figure 10

The group interview was carried out with 3 students working on an extended task. They were given a daily-life problem about shopping, leading to an equation of two variables representing possible prices of two gifts (see Figure 11). We would like to know whether the students can recognize a linear equation in an unfamiliar setting and whether the graphical representation makes sense to them in this context. Findings from this interview are useful for further developing this task to be used in the coming research lesson.

In the first part of the interview, students mainly focused on formulating an equation that can describe the unknown or variable quantities. They also managed to suggest pairs of possible values satisfying the conditions. These pairs of values suggested by students were then pooled together and listed in a spreadsheet projected on a white board. While they agreed that these were solutions for the equations and the points were plotted on the graphing area, they seemed not to recognize that there should be a linear pattern among the points. This was further verified when they were asked to distinguish additional points that may lie on or off the graph. They were not certain whether a point is solution to the formulated equation even when it was plotted and obviously collinear or non-collinear with the previously verified points. The behavior of the students suggests that even if they master the skills of graph plotting and understand the meaning of individual points on the graph, they may not fully understand the overall linear pattern of these points and how a line can represent all possible solutions.



	A	B
1	200	375
2	-266.67	666.67
3	-267	667
4	500	120
5	600	125
6	320	300
7	480	200
8	400	250
9	480	250
10	220	400
11		
12		
13		
14		
15		
16		

Figure 11

Research Lessons

A series of research lessons were designed and tried out in the second term in all secondary 2 classes. These lessons were supposed to be introduction to the concept of linear equation in 2 unknowns and their graphs. There were meetings with mainly the secondary 2 teachers for reviewing the pretest and interviews findings, identifying objects of learning and developing appropriate tasks for these beginning lessons of the chapter. The objects of learning and critical aspects were formulated based on previous findings of students' difficulties and conception. The teaching and learning tasks were designed by analyzing these objects of learning and considering potential of dynamic graphing environment.

As an introduction to the concept of linear equations and graphing, we spend most of these beginning lessons on developing the notions of solutions to an equation in two variables and graphical means to visualize a collection of solutions. Another key idea in the later part is the distinction between linear and non-linear equations. There is also attempt to explain why graphs of linear equations should be straight lines. These emphases are normally not included in this chapter but the school teachers agree that this learning study provides a good opportunity for adding these important elements to the lessons.

<u>Research Lesson 1</u>

Object of learning: recognizing graph of an equation in 2 unknowns in the coordinate plane

Critical aspects:

- Infinitely many points satisfying the equation
- ALL these points form a line or a curve in the coordinate plane
- Equation can be used to determine whether a point lies on the graph

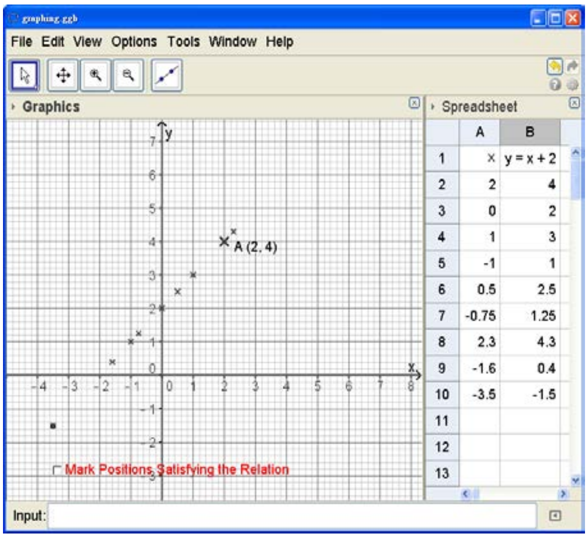
Outline of lesson:

1. Given the point (2,4), ask students to propose different equations satisfied by this point.
2. Find more points satisfying $y=x+2$ and plot them in GeoGebra.
3. Trace other points satisfying the equation in GeoGebra.
4. Define graph as the set of ALL points satisfying the equation.
5. Check whether a point lies on the graph using the equation.

The first lesson starts with a simple revision on the coordinate plane. From examples of points and coordinates, which are worked out by the students, the concept of a relation between coordinates from a point is introduced and explored. Although the students should have experience in plotting and reading points, the idea of relating coordinates from the same point is probably new to them. For example, when focusing on the point (2, 4), we may say that the y-coordinate is twice the x-coordinate. Similarly, students are encouraged to suggest other possible relations to describe this pair of coordinates. When they are comfortable with talking about these relations, a 'short form' can be introduced by teacher to facilitate communication: express them as equations in x and y.

Each of these equations suggested by the students or teacher can give rise to multiple solutions. Choosing $y = x + 2$ as an example, (2, 4) is not the only point where y-coordinate is 2 more than the x-coordinate. The task is then led to search for other possible solutions or points sharing the same property. The process gradually leads to formulation of equations in two variables, solutions and graphs.

In this activity, the use of prepared dynamic graphing tool is crucial. A template is set up so that teacher can work directly on a spreadsheet and graphing area (Figure 12). Points and equations suggested during the activity can be easily recorded and amended on the screen. Various questions are designed to prompt students to identify different points on the grid belonging (or not belonging) to a specific equation, such as points with fractional coordinates, negative coordinates, or lying outside the graphing region that can be easily revealed with adjustment tools built in the software. Finally, automatic marking of suitable positions as a point being freely dragged on the screen gives a vivid holistic picture of the graph as a collection of points satisfying an equation. The dynamic tool specifically designed for the activity greatly enhances the students' participation and teacher's explanation. The teachers admit that the tool is new to them and appreciate its use.



	A	B
1	x	y=x+2
2	2	4
3	0	2
4	1	3
5	-1	1
6	0.5	2.5
7	-0.75	1.25
8	2.3	4.3
9	-1.6	0.4
10	-3.5	-1.5

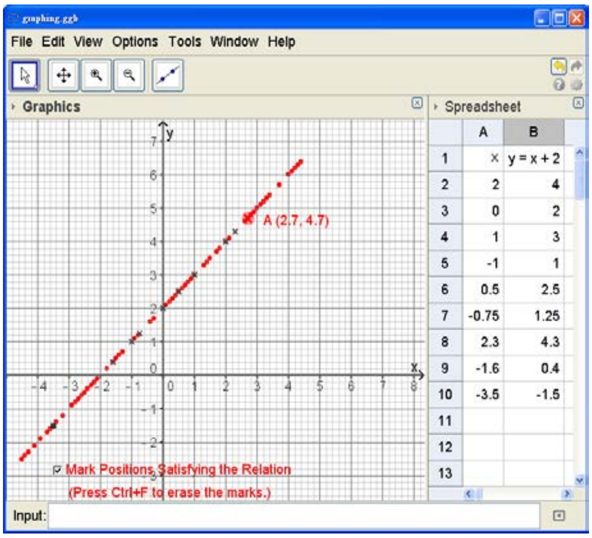


Figure 12

Research Lesson 2

Object of learning: plotting graphs of equations in 2 unknowns

Critical aspects:
- *y* in terms of *x*
- Equidistant points of integral coordinates (equal increment of *x* with equal increment of *y*)

Outline of lesson:

1. Equations are first given in the form $y = f(x)$.
2. Find values of *x* which give integer values of *y*.
3. Observe the "equidistance" between values of *x* and *y*.
4. Observe distance between values of x and the "denominator".
5. Plot the points in GeoGebra.
6. Express *y* in terms of *x* for equations in general form.

The tasks in this lesson focus on more systematic methods of graphing. Building on the previous notion of infinitely possible solutions to an equation and the possibility of representing them in a graph, the students are led to some systematic search of solutions to linear equations and observation of pattern generated by these solutions in the table as well as the coordinate plane. They are suggested to turn a linear equation into the explicit form $y = f(x)$, with their recent skills of change of subject. Once putting in such form, samples of *x* and *y* may be easily computed and recorded in table or coordinate plane (Figure 13).

3. Complete the following table for the equation $y = \frac{2x-1}{3}$ and draw the graph of this equation on the rectangular coordinate plane given.

x	y
–4	
–1	

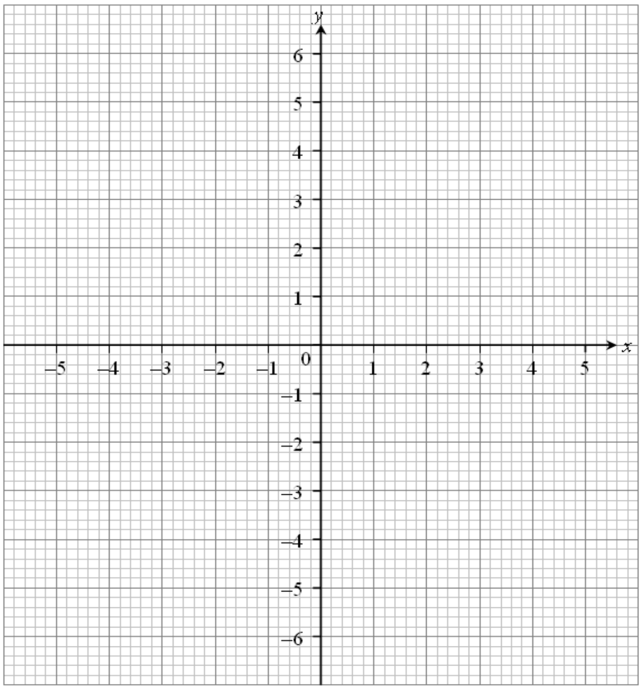


Figure 13

This approach does not merely provide an efficient and reliable method of generating graphs, which is usually emphasized in teaching this unit. Instead, the emphasis is put on noticing spatial and numerical patterns resulting from deliberate choice of equally spaced *x* values. This is intended to suggest why a line graph should result from a linear equation by relating progressions in *x* and *y* values generated in this way. In other words, we attempt to explain linearity of graphs and equations.

<u>Research Lesson 3</u>

Object of learning: linearity of $ax + by + c = 0$

Critical aspects: equation of the graph of a line can be written in the form $ax + by + c = 0$

Outline of lesson:

1. Ask students to guess whether the equation has a linear graph or not without plotting.
2. Check with students by plotting the graph in GeoGebra.
3. Summarize with students the characteristics of equations with linear graphs ($ax + by + c = 0$).

The task in this research lesson helps students to distinguish linear and non-linear graphs and equations through examples and non-examples, some of which generated by students. They are encouraged to articulate forms of linear equations from examples experienced. Once again, the dynamic graphing tool naturally provides a means to quickly and flexibly test and check graphs of any equations imagined by the students or assigned by teacher (Figure 14).

	Equation	The graph of the equation is: a straight line.	The graph of the equation is: not a straight line.
(a)	$2y = x + 1$	✓	
(b)	$x + y = 6$	✓	
(c)	$2x + 3y - 5 = 0$	✓	
(d)	$y = x^2$		✓
(e)	$xy = 1$		✓
(f)	$x^2 + y^2 = 1$		

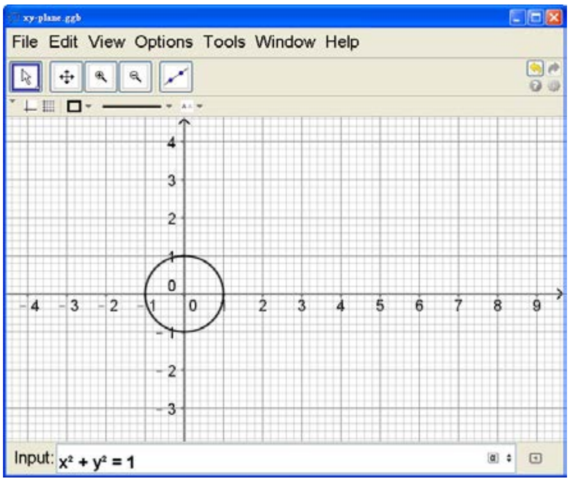


Figure 14

<u>Research Lesson 4</u>

Object of learning: write the equation of a straight line from its graph ($y = mx + c$, m and c are integers)

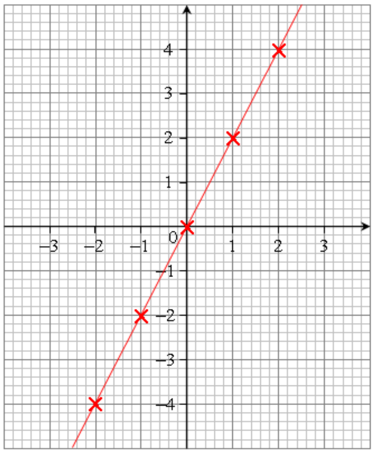
Critical aspects: relationship between x and y coordinates of points on the graph

Outline of lesson:

1. Identify integral points from the graph.
2. List their x, y coordinates in a vertical table.
3. Observe number patterns from $y = mx$ to $y = mx + c$.

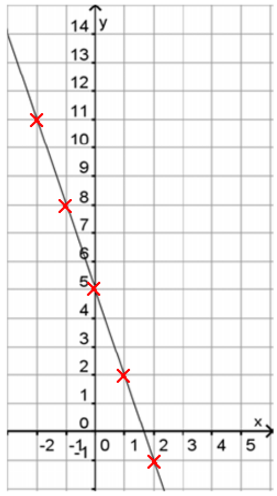
The task in the last research lesson is an enrichment task for those classes satisfactorily completing the previous lessons. It requires students to suggest equation for set of coordinates equally spaced along a line by looking at the pattern of values of x and y in a vertical table form (Figure 15). This task aims at helping the students to gain a sense of linearity from another way, that is, formulating equations from tables of values. We do not aim at systematically finding equations of straight lines which should be covered in Key Stage 4.

x	y
−2	−4
−1	−2
0	0
1	2
2	4



What is the equation of the graph?

Answer: $y = 2x$



x	y
−2	11
−1	8
0	5
1	2
2	−1

What is the equation of the graph?

Answer: $y = -3x + 5$

Figure 15

Evaluation and Reflection

The research lessons were carried out in all secondary two classes in the second semester of the school year 2010-2011. Most of the teachers involved and advisers observed the lessons and conducted post-lesson evaluations. Although there were comments on lesson design and implementation during evaluation after each lesson, the teachers preferred following more or less the same plan throughout the study, without any significant modification.

Most of the tasks could be completed satisfactorily although students' interaction varied from class to class. Teachers found the dynamic graphing tool useful and manageable. They appreciated the new approach, supported by technology, to vividly illustrate and discuss about various aspects of graphing and equations.

There were evaluation of students' performance and understanding through lesson observation, post-lesson written tests for all students and task-based interviews of selected students. It is found that the students could generally acquire the concept of equations and graphs, and demonstrate good skills in various tasks of graphing and solving equations. The basic concept and technique of matching points/solutions and equations were adequately developed for most students.　Regarding understanding of linearity, a range of interesting responses could be identified from the interviews and post-tests. Here are some observations.

- Most students could perceive linearity as “equidistance” of integer points in the graph.
- Expressing y in terms of x, the vertical table form and the “pattern” of integer points help students to find ordered pairs.
- Many students could determine whether the graph is linear or not from the equation, although they could not use the term “degree”, nor could explain why.
- Some weak students avoided to express y in terms of x.
- Some students still could not determine linearity of graphs from equations, but expressing y in terms of x may help students to recognize linearity of data.
- When asking whether the graph or the equation $y = 2x$ is linear or not, one student said no by writing it as $y/x = 2$, but then said yes by writing it as $2x - y = 0$.

Overall, although some students cannot explain how they determine linearity of equations and graphs and may rely on superficial clues in the forms of equations for recognizing linearity, the explicit form of equation, spatial and numerical patterns in square grids and tables are promising means to develop this sense of linearity.　It is hoped that the case reported in this article could let us understand more on students’ learning difficulties, and could provide some useful insights into some effective pedagogical practices on this topic.

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